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A subset of Caffarelli–Kohn–Nirenberg inequalities in the hyperbolic space $\mathbb{H}^N$

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Abstract We prove a subset of inequalities of Caffarelli–Kohn–Nirenberg type in the hyperbolic space $\mathbb{H}^N$, $N \geq 2$, based on invariance with respect to a certain nonlinear scaling group, and study existence of corresponding minimizers. Earlier results concerning the Moser–Trudinger inequality are now interpreted in terms of CKN inequalities on the Poincaré disk.

Keywords Scale invariance · CKN inequalities · Concentration compactness · Weak convergence · Hyperbolic space · Poincaré ball · Hardy inequalities

Mathematics Subject Classification 35J20 · 35J61 · 35J75 · 58J05 · 46B50

1 Introduction

In this paper, we study inequalities that define the embedding for the (homogeneous) Sobolev space $H^1(\mathbb{H}^N)$ of the hyperbolic space $\mathbb{H}^N$, $N \geq 2$, into functional spaces of Lebesgue and Lorentz type, including their radial counterpart, with the radiality understood in terms of the Poincaré ball model. The space $H^1(\mathbb{H}^N)$ is defined as the completion of $C_c^\infty(\mathbb{H}^N)$ with respect to the quadratic form of the Laplace–Beltrami operator on $\mathbb{H}^N$ (see below for details). Similar inequalities in the Euclidean case are a subset of the celebrated Caffarelli–Kohn–Nirenberg (CKN for short) inequalities (see [17], Theorem 4.1, or [10]) and we refer the reader to the paper of Dolbeault, Esteban and Loss [15] for recent results concerning sharp

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estimates of constants and radiality of minimizers). CKN inequalities are distinguished by various optimality properties, including scaling invariance.

1.1 Hyperbolic scaling invariance in the two-dimensional case

The case $N = 2$ of the present paper, once one identifies the space $H_0^1(B)$ of the open unit disk $B \subset \mathbb{R}^2$ as the Sobolev space of the Poincaré disk model of $\mathbb{H}^2$, has been already studied in the paper [3]. The scale-invariant inequalities in [3] provide bounds for appropriate weighted L^p-norms of a function, or its spherical decreasing rearrangement, by the L^N-norm of its gradient on the N-dimensional ball. The inequalities for general values $p \geq N$ are derived, without losing scaling invariance, from the corresponding inequalities for $p = N$ and $p = \infty$ by means of Hölder inequality. For the case $N = 2$, which is considered in the present paper, we have the Leray inequality ([18]) for the case $p = 2$ and the pointwise estimate for radial functions (which also implies the Trudinger inequality for general functions) for the case $p = \infty$. Leray inequality

$$\int_B |\nabla u|^2 dx \geq \frac{1}{4} \int_B \frac{u^2}{r^2 \log\left(\frac{1}{r}\right)^2} dx \tag{1}$$

is a two-dimensional counterpart of the Hardy inequality on $\mathbb{R}^N$, $N \geq 3$

$$Q(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{r^2} dx \geq 0, \tag{2}$$

and, understood as an inequality on the Poincaré disk, has the following invariant form on $\mathbb{H}^2$:

$$\int_{\mathbb{H}^2} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}V_{\mathbb{H}} \geq \frac{1}{16} \int_{\mathbb{H}^2} \frac{u^2}{r^2} \frac{(1-r^2)^2}{\log\left(\frac{1}{r}\right)^2} \mathrm{d}V_{\mathbb{H}}. \tag{3}$$

For $p = \infty$, there is a well-known pointwise estimate for radial functions

$$\sup_{0<r<1} \frac{2\pi u^2(r)}{\log\left(\frac{1}{r}\right)} \leq \int_B |\nabla u|^2 dx \quad = \int_{\mathbb{H}^2} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}V_{\mathbb{H}}, \tag{4}$$

which is a counterpart of the Strauss estimate [26] for radial functions on $\mathbb{R}^N$, $N \geq 2$:

$$\sup_{0<r<1} r^{N-2} u(r)^2 \leq C_N \int_{\mathbb{R}^N} |\nabla u|^2 dx. \tag{5}$$

1.2 Nonlinear scalings for Laplace–Beltrami operators by levels of fundamental solution

Similarly to the original CKN inequalities, which are invariant (up to a normalization factor) with respect to linear scalings $\{u(x) \mapsto u(tx)\}_{t>0}$, their counterparts in [3], in restriction to radial functions, are invariant up to normalization with respect to nonlinear scalings $\{u(r) \mapsto u(r^s)\}_{s>0}$. In this paper, we show that this transformation is a particular case of the scaling transformation $r \mapsto G^{-1}(\lambda G(r))$ where $G(r)$ is the radial fundamental solution (which can be obviously taken here up to an arbitrary scalar multiple) of the Poisson equation for the hyperbolic Laplace–Beltrami operator. In particular, $G(r) = \log \frac{1}{r}$ for the Dirichlet Laplacian on the unit disk, and thus, in the Poincaré disk coordinates, also for the Laplace–Beltrami operator on $\mathbb{H}^2$. The fundamental solution $G(r) = \frac{1}{r^{N-2}}$ plays a similar role in the Euclidean case for $N \geq 3$: with this choice of G the formula $r \mapsto G^{-1}(\lambda G(r))$ means multiplication of r by a power of λ.

The role of fundamental solution in the scaling invariance and furthermore, the role of its square root as a generalized ground state, which we discuss below, is partly motivated by the paper of Adimurthi and Sekar [1].

1.3 Square root of fundamental solution as a ground state: case $p = 2$

It is well known that the Hardy inequality (2) has no minimizer, but any sequence (u_n), that minimizes the quadratic form $Q(u)$ in (2) under a constraint $\int_K u = 1$, where $K \subset \mathbb{R}^N \setminus \{0\}$ is an open relatively compact set, converges in $H^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ to the unique (up to a constant multiple) positive solution $\sqrt{G(r)}$, with $G(r) = \frac{1}{r^{N-2}}$, of the corresponding Euler–Lagrange equation, called the generalized ground state, or virtual bound state. By the ground state alternative of [22], Theorem 1.5 (see also [23], Theorem 1.6), existence of the virtual bound state implies that there is no nonzero nonnegative measurable function W such that $Q(u) \geq \int W u^2$, i.e., the Hardy potential is optimal. A general result in [14] states that, under general conditions on the elliptic operator, the square root of the *positive minimal Green function* is always a generalized ground state. For the sake of consistency of the paper, instead of applying definitions and quoting the exact statement from [14], we give a short direct proof that $\sqrt{G}$ is a generalized ground state in our case. This not only provides the best constant in the hyperbolic counterpart of the Hardy inequality, but also assures that the potential in it cannot be improved.

1.4 Non-radial case with $p \in (2, \frac{2N}{N-2}]$

Let the critical Sobolev exponent be denoted as

$$2^* = \begin{cases} \frac{2N}{N-2}, & N \geq 3, \\ \infty & N = 2. \end{cases}$$

Since there is no embedding of $H^1_{0,\mathrm{loc}}(B \setminus \{0\})$ into $L^p_{\mathrm{loc}}(B \setminus \{0\})$ when $N \geq 3$ and $p > 2^*$, or when $N = 2$ and $p = \infty$, hyperbolic counterparts of general (non-radial) CKN inequalities have to account for this limitation. In particular, when $N = 2$, we have a critical ($p = \infty$) embedding in the form

$$\sup_{0<r<1} \frac{2\pi u^\#(r)^2}{\log\left(\frac{1}{r}\right)} \leq \int_{\mathbb{H}^2} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}V_{\mathbb{H}}, \tag{6}$$

where $u^\#$ is the Schwarz (spherical decreasing) rearrangement of u, relative to the Lebesgue measure on B, derived from (4) and the Polia–Szegö inequality (since the latter holds also with respect to rearrangements relative to the Riemannian measure on $\mathbb{H}^2$ ([6]), inequality (6) holds also when the rearrangement $u^\#$ is defined relative to that measure). Note that the left-hand side is stronger than $\sup_{0<r<1} \frac{u^\#(r)}{\sqrt{\log \frac{e}{r}}}$, which is a quasinorm on the standard Zygmund scale, and is known (see, e.g., [7]) to be equivalent to the Orlicz norm of the functional of critical growth $\int_B e^{au^2} dx$ for the Sobolev space $H^1_0(B)$.

When $N \geq 3$ and $p \in [2, 2^*]$, embeddings of the space $\dot{H}^1(\mathbb{R}^N)$ (the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm $\|\nabla \cdot\|_2$) into weighted L^p-spaces in the non-radial case of CKN inequalities follow from those in the radial case by means of standard rearrangement argument, namely the Polia–Szegö inequality and the Hardy–Littlewood inequality. The latter, however, applies only when the weight in the Lebesgue integral is non-increasing, which is the case only if $p \leq 2^*$. Following the similar approach in the hyperbolic case, we do not have the decresing weight, but instead, for $N \geq 3$, $p \in [2, 2^*]$, instead of embeddings of $H^1(\mathbb{H}^N)$ into weighted L^p-spaces, at embeddings into certain rearrangement-invariant quasi-Banach spaces, which we then identify as intersections of Lorentz spaces $L^{2,p}(\mathbb{H}^N) \cap L^{2^*,p}(\mathbb{H}^N)$ (in the case $N = 2$ we have an intersection of spaces of Zygmund–Lorentz type). These intersections are strictly smaller than L^p, and thus these embeddings refine the embedding of $H^1(\mathbb{H}^N)$ into $L^p(\mathbb{H}^N)$ from [19] (Note that in the Euclidean case there are no embeddings $\dot{H}^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $p \neq 2^*$.)

Scale-invariant inequalities of the present paper follow several other previously established inequalities of Sobolev type on the hyperbolic space. In particular, we would like to mention the Poincaré-Sobolev inequality of Mancini and Sandeep ([19], (1.2), which, as they have shown, by writing it in the half-space coordinates, follows from the Sobolev–Hardy–Mazy'a inequality, which is in turn equivalent to a subset of the original CKN inequalities by means of the ground state transform, also known as Picone identity); as well as related inequalities in [8] and [9]. Inequalities with weight play an important role in the study of Hénon-type equations in hyperbolic space, and a few such embeddings have been developed in [11] and [16]. The scale-invariant inequalities that we prove are significantly sharper than some of those found in literature. In particular, (30) is stronger than (1.1) in [19], while the weight in the embedding in [11], Lemma 2, case $\alpha = 0$, which uses hyperbolic distance from the origin $d(x) = \log \frac{1+r}{1-r}$, behaves as a positive power of r at the origin and as a negative power of $|\log(1-r)|$ at $r = 1$, while the weight (12) in our embeddings (31) and (26) has the power singularity both at the origin and at $r = 1$, see (19)–(20).

The paper is organized as follows. In Sect. 2 we recall definitions related to the hyperbolic space. In Sect. 3 state the main results. In Sect. 4 we prove the inequalities of CKN type. In Sect. 5 we prove existence of minimizers in the hyperbolic CKN inequalities and study related compactness issues. A refined analysis of concentration compactness phenomena, in the form of profile decomposition, is given for sequences of radial functions. In Appendix, we provide cross-references between the results of this paper and the results in [3] for the case $N = 2$, one of them being representation of the Moser–Trudinger inequality as a two-dimensional case of the Sobolev embedding for the hyperbolic space.

2 Preliminaries

2.1 Poincaré ball

Poincaré ball model (coordinate map) of the hyperbolic space $\mathbb{H}^N$, $N \geq 2$, is the unit ball B in $\mathbb{R}^N$ centered at the origin and equipped with the metric

$$ds^2 = \frac{4\sum_{i=1}^N dx_i^2}{(1-r^2)^2}.$$

Here and in what follows the notation r refers to $\sqrt{\sum_{i=1}^N x_i^2}$, the Euclidean distance of a point $x \in B$ from the origin.

The Riemannian measure and the Laplace–Beltrami operators in the Poincaré ball model are, respectively,

$$
dV_{\mathbb{H}} = \frac{2^N}{(1-r^2)^N} dx, \tag{7}
$$

and

$$
\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u|^2 dV_{\mathbb{H}} = \int_B |\nabla u|^2 \frac{2^{N-2} dx}{(1-r^2)^{N-2}}, \tag{8}
$$

where $\nabla_{\mathbb{H}} = \left(\frac{(1-r^2)}{2}\right)^2 \nabla$ and $|\nabla_{\mathbb{H}} u|^2 = \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} u \rangle$ where $\langle , \rangle$ denotes the inner product given by the metric.

Notation $\|u\|_p$ will refer to the $L^p(\mathbb{H}^N, dV_{\mathbb{H}})$-norms. Norms with weight W relative to the measure on $\mathbb{H}^N$ for the spaces $L^p(\mathbb{H}^N, W dV_{\mathbb{H}})$, $1 \leq p < \infty$, be denoted as $\|u\|_{p,W}$. Reference to the weight W when $W = 1$ will be omitted from notation in some instances. Notation $\|u\|_{\infty,W}$ will refer to the supremum norm for the product $|u(x)W(x)|$, and the corresponding space will be denoted as $L^\infty(\mathbb{H}^N, W)$.

The Sobolev space $H^1(\mathbb{H}^N)$ is defined as a completion of C_c^∞ in the norm defined by the quadratic form above. By $H_r^1(\mathbb{H}^N)$ we will denote the subspace of radially symmetric functions of $H^1(\mathbb{H}^N)$ (which is the same as functions in $\mathbb{H}^N$ which are radial with respect to the hyperbolic distance from 0.). We will denote $u \in H_r^1(\mathbb{H}^N)$ by its radial representative $u : [0, 1) \to \mathbb{R}$.
We will denote by ω_{N-1} the surface measure of the unit sphere $S^{N-1} \subset \mathbb{R}^N$.

2.2 Scaling by fundamental solution

Let $f(r) = \frac{(1-r^2)^{N-2}}{r^{N-1}}$ and let $G(r) = \int_r^1 f(t)dt$. It is known that $\frac{1}{N\omega_{N-1}} G$ is a fundamental solution of the hyperbolic Laplacian (see, e.g., [25], Section 3.2). Given that $r \mapsto G(r)$ is a monotone function and the range of G on $\mathbb{H}^N$ is $(0, \infty)$ we define the following multiplicative transformation group on $H_r^1(\mathbb{H}^N)$ by means of the change of radial variable in the Poincaré ball coordinates:

$$
\rho_t(r) = G^{-1}(tG(r)), \; r \in (0, 1), \; t > 0. \tag{9}
$$

The radial map (9) is an analog of the linear scaling in $\mathbb{R}^N$, $N \geq 3$, $\rho_t(r) = \lambda r$, $\lambda = t^{-\frac{1}{N-2}}$, which has the same form as (9) once one substitutes for G the fundamental solution $\frac{C(N)}{r^{N-2}}$ of the Laplace operator in $\mathbb{R}^N$. Action of the linear scaling on functions on $\mathbb{R}^N$, under suitable normalizations, preserves the right- and the left-hand sides in the original CKN inequalities including the quadratic form of the Laplace operator. Transformation (9) similarly preserves the quadratic form of the Laplace–Beltrami operator, if only in restriction to radial functions, and, furthermore, one can show by elementary computations based on change of variable under the integral that every radial diffeomorphism with this property is necessarily of the form (9).

When $N = 2$, the Laplace operator does not have a positive fundamental solution on the whole on $\mathbb{R}^2$, but the same construction on the unit disk B, using the fundamental solution $G(r) = \frac{1}{2\pi} \log \frac{1}{r}$, defines an automorphism $\rho_t(r) = r^t$ of B, whose action preserves, up to a normalization factor, the quadratic form of the Laplacian on B evaluated on radial functions (see [2,3]). There also exists a family of maps, which we write in the notation of a complex variable as $z \mapsto z^m$, $m \in \mathbb{N}$, whose action preserves, up to normalization, the quadratic form of the Laplacian on B for general functions as well (see [5]). This case is appended to the present paper, via the Poincaré ball model, as the case of $\mathbb{H}^2$.

Indeed, for general N, we also have

Proposition 2.1 *The family of operators*

$$D = \left\{g_t \ : \ g_t u(r) := t^{-\frac{1}{2}} u(\rho_t(r)) = t^{-\frac{1}{2}} u(G^{-1}(tG(r))), t > 0\right\} \tag{10}$$

forms a multiplicative group of isometries on $H_r^1(\mathbb{H}^N)$.

Proof The multiplicative property of the group is obvious. In order to prove that (10) defines an isometry, first consider a general change of the radial variable in the integral $\int_0^1 u_r^2 \frac{dr}{f(r)}$ appearing in the expression of the norm in (8) when u is radial. Let ρ be a general increasing C^1-function that maps [0, 1] bijectively onto itself, then changing the variable as $r = \rho(t)$ and $v(t) = u(r)$ we get

$$\int_0^1 u_r^2 \frac{dr}{f(r)} = \int_0^1 \frac{v_t^2}{\rho'(t)^2} \frac{\rho'(t)}{f(\rho(t))} dt = \int_0^1 \left[\frac{f(t)}{\rho'(t) f(\rho(t))}\right] v_t^2 \frac{dt}{f(t)}. \tag{11}$$

Since the "test weight" u'^2 is arbitrary, the right-hand side is a positive multiple of $\int_0^1 u_r^2 \frac{dr}{f(r)}$ if and only if $\frac{f(t)}{\rho'(t) f(\rho(t))}$ is a positive constant, i.e., $\frac{dG(\rho(r))}{dr} = \lambda \frac{dG(r)}{dr}$ for some $\lambda > 0$. Given that $\rho(1) = 1$ and $G(1) = 0$, this is equivalent to $G(\rho(r)) = \lambda G(r)$. Since G is a diffeomorphism between $(0, 1)$ and $(0, \infty)$, this defines the function ρ as in (9) with the required isometric property, and, by necessity, it is the only radial function with this property. $\square$

Proposition 2.2 *Let* $N \geq 2$ *and let*

$$V_p(r) = \frac{f(r)^2(1-r^2)^2}{G(r)^{\frac{p+2}{2}}} \quad \textit{for } p \in [1, \infty), \ \ V_\infty(r) = \frac{1}{\sqrt{G(r)}}. \tag{12}$$

Then (10) *is an isometry also in* $L^p(\mathbb{H}^N; V_p \mathrm{d}V_{\mathbb{H}})$, $p \in [1, \infty]$.

Proof Similarly to the proof of Proposition 2.1, we will use the change of variable (this time already fixed as (9)) in order to derive the weight V_p as a unique one (up to a multiple) for which $\int_0^1 |u(r)|^p V_p(r) \frac{dr}{f(r)}$ is invariant with respect to (10). Consider first $p < \infty$. Taking into account that, as in the proof of Proposition 2.1, $G(\rho_t(r)) = tG(r)$ and, consequently, $f(\rho_t(r))\rho_t'(r) = tf(r)$, we have, omitting the subscript t,

$$\int_0^1 t^{-\frac{p}{2}} |u|^p V_p(\rho) \frac{d\rho}{f(\rho)} = \int_0^1 t^{-\frac{p}{2}} |u|^p V_p(\rho) \frac{\rho'(r) dr}{f(\rho)} = \int_0^1 t^{-\frac{p}{2}} |u|^p V_p(\rho) \frac{tf(r)dr}{f(\rho)}. \tag{13}$$

It remains to substitute $t = G(\rho)/G(r)$ and equate the integrands as in the proof of Proposition 2.1 to arrive, after elementary computations, at an expression for $V_p(\rho)/V_p(r)$ that gives (12) up to a constant.

If $p = \infty$, we have

$$\sup_{\rho<1} \frac{|u(\rho)|}{\sqrt{G(\rho)}} = \frac{|u(\rho)|}{\sqrt{tG(r)}} = \sup_{r<1} \frac{|t^{-1/2} u(\rho(r))|}{\sqrt{G(r)}}.$$

$\square$

The following statement also follows by direct computation.

Let us give some exact and some asymptotic values for G and V_p.

$$G(r) = \log \frac{1}{r} \quad \text{for } N = 2, \tag{14}$$

$$G(r) = \frac{(1-r)^2}{r} \quad \text{for } N = 3, \tag{15}$$

$$G(r) = \frac{C(N)}{r^{N-2}}(1 + o_{r\to 0}(1)) \quad \text{for } N \geq 3, \tag{16}$$

$$G(r) = C(N)(1-r)^{N-1}(1 + o_{r\to 1}(1)) \quad \text{for } N \geq 3. \tag{17}$$

$$V_p(r) = \frac{(1-r^2)^2}{r^2(\log \frac{1}{r})^{\frac{p+2}{2}}} \quad \text{for } N = 2,\, 1 < p < \infty, \tag{18}$$

$$V_p(r) = \frac{C(N,p)}{r^{N(1-p/2^*)}}(1 + o_{r\to 0}(1)) \quad \text{for } N \geq 3,\, 1 < p < \infty, \tag{19}$$

$$V_p(r) = \frac{C(N,p)}{(1-r)^{\frac{(N-1)(p-2)}{2}}}(1 + o_{r\to 1}(1)) \quad \text{for } N \geq 3,\, 1 < p < \infty. \tag{20}$$

2.3 Lorentz spaces involved in the estimates

Lorentz spaces $L^{p,q}$ for a measure space, and in the present paper for $(\mathbb{H}^N, \mathrm{d}V_{\mathbb{H}})$, are complete linear quasinormed vector spaces of measurable functions such that

$$\|u\|_{p,q} = \left(\int_0^\infty \left(t^{1/p} u^*(t)\right)^q \frac{dt}{t} \right)^{1/q}, q < \infty. \tag{21}$$

where $u^* : [0,\infty) \to [0,\infty$ is the symmetric decreasing rearrangement of u. We recall that $L^{p,p}$ coincides with the Lebesgue space L^p.
We define the Schwarz rearrangement of a measurable function $u : \mathbb{H}^N \to \mathbb{R}$, denoted $u^\# : \mathbb{H}^N \to [0,\infty)$, as the radial function given, in the Poincaré ball coordinates by

$$u^\#(x) = u^*(V_{\mathbb{H}}(B_{|x|})), \tag{22}$$

where $V_{\mathbb{H}}$ is the Riemannian measure in the hyperbolic metric, an $B_{|x|}$ is a Euclidean ball in the Poincaré ball coordinates, of Euclidean radius $|x|$, centered at the origin.

Theorem 2.3 *The set of all measurable functions satisfying*

$$\|u^\#\|_{q,V_q} < \infty, \quad q \in (1,\infty), \tag{23}$$

is a linear space with $\|u^\#\|_{q,V_q}$ as a quasinorm. Furthermore, if $N \geq 3$ and $2 \leq q \leq 2^*$, then this space coincides with the intersection of Lorentz spaces $L^{2,q}(\mathbb{H}^N) \cap L^{2^*,q}(\mathbb{H}^N)$, and quasinorm (23) is equivalent to the intersection quasinorm $\|\cdot\|_{2,q} + \|\cdot\|_{2^*,q}$. Moreover, $L^{2,q}(\mathbb{H}^N) \cap L^{2^*,q}(\mathbb{H}^N) \hookrightarrow L^q(\mathbb{H}^N)$.

Proof 1. Let us prove first that (23) defines a quasinorm, that is, there is a constant C such that for every measurable functions u and v one has

$$\|(u+v)^\#\|_{q,V_q} \leq C\left(\|u^\#\|_{q,V_q} + \|v^\#\|_{q,V_q}\right). \tag{24}$$

The argument is similar to the classical one for Lorentz spaces. We may assume that the right-hand side is finite, since otherwise this is tautologically true. First observe that by a simple change of variable

$$\|(u+v)^{\#}\|_{q,V_q} = \left(\int_0^{\infty} (u+v)^*(t)^q \ \hat{V}_q(t)dt \right)^{1/q}$$

where $\hat{V}_q$ is given by $\hat{V}_q(t)|_{t=V_{\mathbb{H}}(B_r)} := V_q(r)$. By the well-known inequality for rearrangements of functions on a measure space, we have $(u+v)^*(t) \le u^*(\frac{t}{2}) + v^*(\frac{t}{2})$. Using this inequality, (24) follows from elementary estimations and a change of variable once we have the following doubling property for $\hat{V}_q$: There exists a constant $C > 0$ such that

$$\hat{V}_q(t) \le C\hat{V}_q(t/2), \quad t > 0. \tag{25}$$

Note that the function $\hat{V}_q$ is positive and continuous, so that the quotient $\frac{\hat{V}_q(t)}{\hat{V}_q(t/2)}$ is bounded on every compact subset of $(0,\infty)$ (by a constant possibly dependent on the subset). Thus, in order to prove that this quotient is uniformly bounded, it suffices to consider asymptotics of $\hat{V}_q$ at zero and at infinity. For the sake of brevity, we use the notation $f \sim g$ near a given point whenever f/g has a positive limit at that point. Omitting multiplicative constants, we have $t = V_{\mathbb{H}}(B_r) \sim r^N$ at zero and $t \sim (1-r)^{1-N}$ at infinity, so that $r \sim t^{1/N}$ near zero and $1 - r \sim t^{-\frac{1}{N-1}}$ near infinity. Assume first that $N \ge 3$. From (19) we have $\hat{V}_q(t) \sim r^{-N(1-q/2^*)} \sim t^{-(1-q/2^*)}$ near zero. From (20) we have $\hat{V}_q(t) \sim (1-r)^{-\frac{(N-1)(q-2)}{2}} \sim t^{\frac{(q-2)}{2}}$ near infinity. Both expressions satisfy doubling property (25). If $N = 2$, we have by (18) $\hat{V}_q(t) = \frac{(1-r^2)^2}{r^2(\log\frac{1}{r})^{\frac{q+2}{2}}}$. This gives us $\hat{V}_q(t) \sim \frac{1}{t(\log|t|)^{\frac{q+2}{2}}}$ near zero and $\hat{V}_q(t) \sim \frac{1}{t^2(\log t)^{\frac{q+2}{2}}}$ near infinity. Since the function $1/\log|t|$ easily verifies doubling property (25), so does $\hat{V}_q(t)$. We conclude that (23) defines a quasinorm, from which it is immediate that condition (23) defines a linear space.

2. Calculations in the previous step for $N \ge 3$ give that $\hat{V}_q(t) \sim t^{q/2^*-1}$ at zero and is $\hat{V}_q(t) \sim t^{q/2-1}$ at infinity. The exponents in the asymptotics are the same as in the weights for the Lorentz spaces $L^{2^*,q}$ and $L^{2,q}$, respectively. When $2 \le q \le 2^*$, the quantity $t^{q/2^*-1}$ dominates $t^{q/2-1}$ at zero, while $t^{q/2-1}$ dominates $t^{q/2^*-1}$ at infinity, and therefore $C_1(t^{q/2^*-1} + t^{q/2-1}) \le \hat{V}_q(t) \le C_2(t^{q/2^*-1} + t^{q/2-1})$. This proves the second assertion of the theorem.

3. The last assertion of the theorem follows from the fact that the weight $t^{\frac{2}{q}-1} + t^{\frac{2^*}{q}-1}$ is bounded away from zero for $2 < q < 2^*$, which implies that the L^q-norm is bounded by the quasinorm of $L^{2,q} \cap L^{2^*,q}$. □

Remark 2.4 When $N = 2$, an argument similar to the step 2 of the proof above shows the space defined by the quasinorm $\|\cdot\|_{q,V_q}$, $2 \le q \le \infty$, is an intersection of two spaces of Lorentz–Zygmund type. For the sake of brevity, we will denote this space as $\mathcal{L}^q(\mathbb{H}^2)$. When $q = \infty$ these spaces coincide and the quasinorm of $\mathcal{L}^\infty(\mathbb{H}^2)$ is $\sup_{0<r<1} \frac{u^{\#}(r)}{\sqrt{\log\frac{1}{r}}}$.

3 Main results

The main results of this paper are the inequalities below. They all use weights V_p, $2 \le p \le \infty$, defined in (12).

Theorem 3.1 *Let $N \geq 2$. For all $p \in [2, \infty]$ there exists a constant $c(N, p) > 0$, such that for and every $u \in H^1(\mathbb{H}^N)$,*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}V_{\mathbb{H}} \geq c(N, p)\|u^\#\|^2_{p,V_p}. \tag{26}$$

Moreover, for $N \geq 3$ and $p \in [2, 2^*]$, inequality (26) is equivalent to an embedding into the intersection of Lorentz spaces

$$H^1(\mathbb{H}^N) \hookrightarrow L^{2,p}(\mathbb{H}^N) \cap L^{2^*,p}(\mathbb{H}^N) \tag{27}$$

Remark 3.2 From (27) follows the embedding of Mancini and Sandeep in [19]

$$H^1(\mathbb{H}^N) \hookrightarrow L^p(\mathbb{H}^N),\ 2 \leq p \leq 2^*. \tag{28}$$

Furthermore, since Lorentz spaces are monotone increasing with respect to the second index, (27) is in fact equivalent to its case for $p = 2$, namely

$$H^1(\mathbb{H}^N) \hookrightarrow L^2(\mathbb{H}^N) \cap L^{2^*,2}(\mathbb{H}^N). \tag{29}$$

Remark 3.3 To illustrate optimality of the inequality (26), consider its restriction to $H^1_r(\mathbb{H}^N)$. If $V(r)$ is a continuous function and $\frac{V(r)}{V_p(r)} \to +\infty$ when $r \to 0$ or $r \to 1$, then the inequality (26) with V_p replaced by V will be false. Indeed, one can fix any nonzero function $w \in H^1_r(\mathbb{H}^N)$ and, by changing the radial variable under the integral defining the $L^p - norm$, easily find that $\sup_{t>0} \|g_t w\|_{p,V} = \infty$.

Theorem 3.4 *For every $u \in H^1(\mathbb{H}^N)$,*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}V_{\mathbb{H}} \geq \frac{1}{16}\int_{\mathbb{H}^N} |u|^2 V_2 \mathrm{d}V_{\mathbb{H}}, \tag{30}$$

and there is no other measurable function $W \geq V_2$ for which this inequality could hold with W instead of V_2.

Note that for $N = 2$ this is the classical Leray inequality, [18].

Remark 3.5 From inequality (30), one can infer the value $\frac{(N-1)^2}{4}$ for the bottom of the spectrum for the hyperbolic Laplace–Beltrami operator, (1.1) in [19]. Indeed, leaving details to the reader, one verifies first that V_2 is a decreasing function of r, so that $V_2(r) \geq V_2(1)$. From the definition of G we have $G(r) = \frac{2^{N-2}(1-r)^{N-1}}{N-1}(1 + o_{r\to 1}(1))$, and thus

$$V_2(r) = \left(\frac{2f(r)(1-r)}{G(r)}\right)^2 (1 + o_{r\to 1}(1)) = (2(N-1))^2 (1 + o_{r\to 1}(1)),$$

which implies $\frac{V_2(r)}{16} \geq \frac{(N-1)^2}{4}$.

Theorem 3.6 *For $N \geq 2$, $p \in [2, \infty]$, there exists a constant $c_r(N, p) > 0$, such that for every $u \in H^1_r(\mathbb{H}^N)$,*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}V_{\mathbb{H}} \geq c_r(N, p)\|u\|^2_{p,V_p}. \tag{31}$$

The constants $c_r(N, 2) = \frac{1}{16}$ and $c_r(N, \infty) = 2^{N-2}\omega_{N-1}$ are exact, and $c_r(N, p) \leq c_r(N, 2)c_r(N, \infty)^{p-2}$, $2 < p < \infty$.

The case $p = \infty$ has been already proved by Hasegawa ([16], formula (A.3), Lemma A.1.)

We have the following result concerning minimizers of the above inequality.

Theorem 3.7 *Let $N \geq 2$. If $p \in (2, \infty]$, then the minimum in*

$$c_r(N, p) = \inf_{u \in H_r^1(\mathbb{H}^N), \|u\|_{p,V_p}=1} \|\nabla_{\mathbb{H}} u\|_2^2. \tag{32}$$

is attained. Furthermore, for any minimizing sequence (u_n), there exists a renamed subsequence and sequence (t_n) such that $(g_{t_n} u_n)$ converges to a minimizer. When $p = \infty$, the minimizer is unique up to the sign and the action of the group (10), and equals

$$\psi_N(r) = \begin{cases} 1, & 0 \leq r \leq r_1, \\ G(r), & r_1 \leq r \leq 1, \end{cases} \tag{33}$$

where $r_1 = G^{-1}(1)$.

Note that the function (33) is a generalization of the test function used by Moser [21] in the case $N = 2$.

Our further results include Theorem 5.1 on cocompactness of embeddings of $H_r^1(\mathbb{H}^N)$ relative to the group (10), Theorem 5.3 on structure of unbounded sequences in $H_r^1(\mathbb{H}^N)$, Theorem 5.4 on compactness of embeddings of the inhomogeneous counterpart of $H_r^1(\mathbb{H}^N)$ into L^p-spaces, and an elementary Theorem 5.5 on compactness of embeddings in presence of ”sub-Hardy” potentials.

4 Proofs of the inequalities

Let us look in more detail at the weight $V_2(r) = \left(\frac{f(r)(1-r^2)}{G(r)}\right)^2$ for $N \geq 3$. At $r = 1$ the value of V_2 is finite and positive, and near zero $V_2(r) = \frac{(N-2)^2}{r^2}(1 + o_{r\to 0}(1))$. In particular, $V_2(r) = \left(\frac{(1+r)^2}{r}\right)^2$ for $N = 3$. In other words, the weight $\frac{1}{16}V_2$ that appears in the theorem below is qualitatively similar to the weight $\frac{(N-2)^2}{4r^2}$ from the usual radial Hardy inequality in $\mathbb{R}^N$. We are going to use the following well-known identity (see, e.g., [13]).

Lemma 4.1 (Ground state transform). *Let Ω be a domain in $\mathbb{R}^N$ and let V be a continuous function on Ω. Let $A(x)$, $x \in \Omega$, be a symmetric real-valued positive matrix with continuous coefficients. If $v \in C^2(\Omega)$ is a positive solution of the equation (understood in the sense of weak derivatives) $-\nabla \cdot A(x)\nabla v(x) = V(x)v(x)$, $x \in \Omega$, then the following identity holds for any $u \in C_c^\infty(\Omega)$:*

$$\int_\Omega \left(A(x)\nabla u(x) \cdot \nabla u(x) - V(x)|u(x)|^2\right) dx = \int_\Omega v^2 A(x)\nabla\left(\frac{u(x)}{v(x)}\right) \cdot \nabla\left(\frac{u(x)}{v(x)}\right) dx. \tag{34}$$

Proof of Theorem 3.4. Direct calculation of $-\Delta_{\mathbb{H}}\sqrt{G(r)}$ shows that $v(r) = \sqrt{G(r)}$ satisfies the Euler–Lagrange equation for the functional

$$Q(u) := \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}V_{\mathbb{H}} - \frac{1}{16}\int_{\mathbb{H}^N} V_2 |u|^2 \mathrm{d}V_{\mathbb{H}}. \tag{35}$$

Since $v > 0$, we can represent (35) by means of (34) in the form

$$Q(u) = \int_{\mathbb{H}^N} v^2 \left|\nabla_{\mathbb{H}} \left(\frac{u}{v}\right)\right|^2 \mathrm{d}V_{\mathbb{H}} \geq 0. \tag{36}$$

The remaining assertion of the theorem follows from Theorem 1.4 of [22]. Indeed, Theorem 1.4 of [22] states, among the rest, that if the quadratic form as in (34) (in particular our form (35)) has a *null sequence* (Definition 1.1, [22]), it admits no spectral gap. Absence of spectral gap (Definition 1.2, [22]) is exactly the second assertion of our theorem. A null sequence is a sequence (v_n) which converges to a positive solution v of $Q'(v) = 0$ uniformly on compact subsets, and satisfies $Q(v_n) \to 0$. Thus it suffices to construct a null sequence. Let, $r_1 = G^{-1}(1)$, and let

$$\epsilon_n(r) = \begin{cases} -\frac{1}{2n} & \text{for } r \leq r_1, \\ \frac{1}{2n} & \text{for } r \geq r_1, \end{cases} \quad n \in \mathbb{N}.$$

Define now

$$v_n(r) = v(r)^{1+2\epsilon_n(r)}. \tag{37}$$

Note that $v(r_1) = 1$ and therefore the functions v_n are Lipschitz continuous at r_1 and smooth elsewhere, so that $v_n \in H^1_{\mathrm{loc}}(\mathbb{H}^N)$. Then

$$\begin{aligned} Q(v_n) &= \omega_{N-1} \int_0^1 G(r) \left|\partial_r \frac{v_n}{v}\right|^2 \frac{dr}{f(r)} \\ &= \omega_{N-1} \int_0^1 G(r) |\partial_r G(r)^{\epsilon_n(r)}|^2 \frac{dr}{f(r)} \\ &= \int_0^1 \epsilon_n(r)^2 \frac{f(r)}{G(r)^{1+2\epsilon_n(r)}} dr \\ &= -\frac{1}{n^2} \int_0^1 \frac{1}{2\epsilon_n(r)} \partial_r G(r)^{-2\epsilon_n(r)} dr \\ &= \frac{1}{n} \int_0^{r_1} \partial_r [G(r)^{1/n}] dr - \frac{1}{n} \int_{r_1}^1 \partial_r [G(r)^{-1/n}] dr \\ &= \frac{1}{n} G(r)^{1/n}|_0^{r_1} - \frac{1}{n} G(r)^{-1/n}|_{r_1}^1 = \frac{2G(r_1)}{n} = \frac{2}{n} \to 0. \end{aligned}$$

At the last step of calculation we use the asymptotic values of G at 0 and 1 provided at the end of Sect. 2.2 that yield $G(r)^{-2\epsilon_n(r)} \to 0$ as $r \to 0$ or $r \to 1$. □

Proof of Theorem 3.6. Let us first prove (31) for the case $p = \infty$, that is,

$$\frac{|u(r)|}{\sqrt{G(r)}} \leq \frac{1}{\sqrt{\omega_{N-1}}} \|\nabla_{\mathbb{H}} u\|_2, \ u \in H^1_r(\mathbb{H}^N). \tag{38}$$

We include this proof only for the sake of consistency, as an essentially the same argument is found in the proof of Lemma A1, [16]. Using Cauchy–Schwarz inequality we have

$$u(r) = \int_1^r u'(s) ds \leq \int_1^r \frac{u'(s)}{\sqrt{f(s)}} \sqrt{f(s)} ds \tag{39}$$

$$\leq \left(\int_0^1 \frac{u'(s)^2}{f(s)} ds\right)^{\frac{1}{2}} \left(\int_r^1 f(s) ds\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2^{N-2}\omega_{N-1}}} \|\nabla_{\mathbb{H}} u\|_2 \sqrt{G(r)}, \tag{40}$$

which verifies (38). Inequality (31) for $p \in (2, \infty)$, including the asserted estimate for $c_r(N, p)$, follows now immediately from application of Hölder inequality to (35) and (38).

The exact constant $c_r(N, 2)$ is realized on the normalized sequence v_n from (37). An easy computation shows that $\|\nabla_{\mathbb{H}} v_n\|_2 \to \infty$ and thus $Q(w_n) \to 0$ where $w_n = v_n/\|\nabla_{\mathbb{H}} v_n\|_2$. This immediately implies that $\frac{1}{16}\int_{\mathbb{H}^N} |w_n|^2 V_2 \mathrm{d}V_{\mathbb{H}} = \|\nabla_{\mathbb{H}} v_n\|_2^2 - Q(w_n) \to 1$.

The exact constant $c_r(N, \infty)$ is realized on the Lipschitz continuous function (33). Indeed,

$$
\begin{aligned}
\|\nabla_{\mathbb{H}} \psi_N\|_2^2 &= \omega_{N-1} 2^{N-2} \int_{r_1}^{1} |G'(r)|^2 \frac{dr}{f(r)} \\
&= \omega_{N-1} 2^{N-2} \int_{r_1}^{1} f(r) dr = \omega_{N-1} 2^{N-2} G(r_1) = \omega_{N-1} 2^{N-2}.
\end{aligned}
$$

At the same time $\psi_N(r_1)^2/G(r_1) = 1$, which proves that the constant $c_r(N, \infty)$ is exact. □

Proof of Theorem 3.1. Inequality (26) follows from (31) once we take into account the Polya-Szegö inequality for rearrangements in the hyperbolic space, $\|\nabla_{\mathbb{H}} u^{\#}\|_2 \le \|\nabla_{\mathbb{H}} u\|_2$ (see [6]). Embedding of $H^1(\mathbb{H}^N)$ into $L^{2,p} \cap L^{2^*,p}$ follows from Theorem 2.3. □

5 Cocompactness, profile decomposition, and minimizers

In this section we follow the framework of [3] (which, implicitly, studied the case $N = 2$ of the present paper).

We recall that an embedding of a Hilbert space H into a Banach space Y is called *cocompact relative to a group of unitary operators D* if any sequence $(u_n) \subset H$ D-weakly convergent to zero (i.e., is such that for any $(g_n) \subset D$, $g_n u_n \rightharpoonup 0$), converges in the norm of Y.

Embeddings of $H_r^1(\mathbb{H}^N)$ defined by (31) are cocompact relative to the following discrete subgroup of the group (10):

$$
D_0 = \left\{ g_j \in D : \ j \in 2^{\mathbb{Z}} \right\}. \tag{41}
$$

Theorem 5.1 *Let $N \ge 2$. For any $p \in (2, \infty]$, the embedding $H_r^1(\mathbb{H}^N) \hookrightarrow L^p(\mathbb{H}^N; V_p \mathrm{d}V_{\mathbb{H}})$ is cocompact relative to the group D_0.*

The argument is an elementary generalization of the proof for the case $N = 2$ in [3], Lemma 3.3, Lemma 3.4 and their interpretation in Remark 3.5. The main step in the proof of the theorem is the case $p = \infty$ which is a trivial generalization of [3], Lemmas 3.3.

Proposition 5.2 *The embedding $H_r^1(\mathbb{H}^N) \hookrightarrow L^\infty(\mathbb{H}^N, V_\infty)$ is cocompact relative to the group* (10).

Proof Let $(u_n) \subset H_r^1(\mathbb{H}^N)$ such that for any sequence (t_n) of positive numbers, $g_{t_n} u_n \rightharpoonup 0$. Let us fix the sequence (s_n) such that

$$
\frac{|u_n(s_n)|}{\sqrt{G(s_n)}} \ge \frac{1}{2} \sup_{0<r<1} \frac{|u_n(r)|}{\sqrt{G(r)}}.
$$

Note that, setting $r_1 = G^{-1}(1)$, we can write the left-hand side of the expression above as $|g_{t_n} u_n(r_1)|$ where $t_n = G(s_n)$. Since $g_{t_n} u_n \rightharpoonup 0$, $g_{t_n} u_n(r_1) \to 0$ by local compactness of Sobolev embeddings in dimension 1, and the proposition follows. □

We can now prove Theorem 5.1.

Proof Let $(u_n) \subset H_r^1(\mathbb{H}^N)$ such that for any sequence of numbers $j_n \in 2^{\mathbb{Z}}$, $g_{j_n} u_n \rightharpoonup 0$. Since for any sequence (t_n) of positive numbers there exist $j_n \in 2^{\mathbb{Z}}$ such that $1 \le t_n/j_n \le 2$, we have that $g_{t_n} u_n \rightharpoonup 0$ for any (t_n). Indeed, assume the contrary, namely that on a renamed subsequence $g_{t_n} u_n \rightharpoonup w \neq 0$. Extracting a further subsequence such that $t_n/j_n \to a \in [1, 2]$, taking into account that a multiplicative group of isometries on Hilbert space satisfies $g_s^* = g_s^{-1} = g_{1/s}$, and using the general notation of scalar product for the one of $H_r^1(\mathbb{H}^N)$, we have

$$
\begin{aligned}
|(g_{t_n} u_n, w)| &= |(g_{t_n/j_n} g_{j_n} u_n, w)| = |(g_{j_n} u_n, g_{j_n/t_n} w)| \\
&\le |(g_{j_n} u_n, g_{j_n/t_n} w - g_{1/a} w)| + |(g_{j_n} u_n, g_{1/a} w)| \\
&\le |\|\nabla_{\mathbb{H}} u_n\|_2 \|\nabla_{\mathbb{H}} (g_{j_n/t_n} - g_{1/a}) w\|_2 + |(g_{j_n} u_n, g_{1/a} w)| \to 0,
\end{aligned}
$$

while the left-hand side converges to $\|\nabla_{\mathbb{H}} w\|_2^2 \neq 0$, a contradiction.

Consequently,

$$
\|u_n\|_{p,V_p}^p \le \|u_n\|_{2,V_2}^2 \|u_n\|_{\infty,V_\infty}^{p-2} \le C \|\nabla_{\mathbb{H}} u_n\|_2^2 \|u_n\|_{\infty,V_\infty}^{p-2} \to 0
$$

by Proposition 5.2. □

As a consequence of Theorem 5.1, we have the following structural result for general bounded sequences in $H_r^1(\mathbb{H}^N)$.

Theorem 5.3 *Let $u_k \rightharpoonup 0$ in $H_r^1(\mathbb{H}^N)$, $N \ge 2$. There exist sequences $(j_k^{(n)})_{k\in\mathbb{N}} \subset 2^{\mathbb{Z}}$, $n \in \mathbb{N}$, such that for a renumbered subsequence of (u_k),*

$$
g_{1/j_k^{(n)}} u_k \rightharpoonup w^{(n)}, \tag{42}
$$

$$
\left|\log_2 j_k^{(m)} - \log_2 j_k^{(n)}\right| \to \infty \quad \text{for } n \neq m, \tag{43}
$$

$$
\sum_{n\in\mathbb{N}} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} w^{(n)}|^2 \mathrm{d}V_{\mathbb{H}} \le \limsup \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}} u_k|^2 \mathrm{d}V_{\mathbb{H}}, \tag{44}
$$

$$
u_k - \sum_{n\in\mathbb{N}} g_{j_k^{(n)}} w^{(n)} \to 0 \text{ in } L^p(\mathbb{H}^N, V_p \mathrm{d}V_{\mathbb{H}}),\ p \in (2, \infty], \tag{45}
$$

and the series $\sum_{n\in\mathbb{N}} g_{j_k^{(n)}} w^{(n)}$ converges in $H^1(\mathbb{H}^N)$ unconditionally and uniformly in k.

Proof This theorem is just a particular case of the theorem on profile decompositions in general Hilbert space, Theorem 3.1 in [27], for $H^1(\mathbb{H}^N)$ equipped with the group D (the unconditional convergence has been explicitly stated only later, in a more general result, Theorem 5.5 (see Definition 2.5) in [24]). For more details about application of the general theorem to the group D, we refer the reader to [3]. □

We can now prove Theorem 3.7.

Proof 1. Let (u_n) be a minimizing sequence for (32), that is, $\|u_n\|_{p,V_p} = 1$ and $\|\nabla_{\mathbb{H}} u_n\|_2^2 \to c_r(p)$. Since both norms in (32) are D-invariant, for any sequence $j_n \in 2^{\mathbb{Z}}$, $(g_{j_n} u_n)$ is also a minimizing sequence. Without loss of generality, we may assume that (u_n) has a weak limit $u \neq 0$. Indeed, if $(g_{j_n} u_n) \rightharpoonup 0$ for any $j_n \in 2^{\mathbb{Z}}$, then by Theorem 5.1 $u_n \to 0$ in $L^p(\mathbb{H}^N, V_p \mathrm{d}V_{\mathbb{H}})$, which is a contradiction. We may then pass to a subsequence of $(g_{j_n} u_n)$ that has a nonzero weak limit and rename it as u_n.

2. Assume now that $p < \infty$. Let $v_n = u_n - u$. By Brezis-Lieb Lemma and by the well-known elementary relation for square norms in the Hilbert space, $\|u_n\|^2 - \|u\|^2 - \|u_n - u\|^2 \to 0$, we have (applying in the second relation the definition (32) of $c_r(N, p)$):

$$\begin{aligned}
&\|u\|_{p,V_p}^p + \|v_n\|_{p,V_p}^p \to \|u_n\|_{p,V_p}^p = 1,\\
&\|u\|_{p,V_p}^2 + \|v_n\|_{p,V_p}^2\\
&\quad \le \frac{1}{c_r(N,p)}\|\nabla_{\mathbb{H}} u\|_2^2 + \frac{1}{c_r(N,p)}\|\nabla_{\mathbb{H}} v_n\|_2^2\\
&\quad = \frac{1}{c_r(N,p)}\|\nabla_{\mathbb{H}} u_n\|_2^2 + o(1) \to 1.
\end{aligned}$$

Given that $p > 2$ and $u \neq 0$, two relations above can hold simultaneously only if $v_n \to 0$ in $L^p(\mathbb{H}^N, V_p \mathrm{d}V_{\mathbb{H}})$ or if $u = 0$ (which is ruled out on the step 1). Then $\|u\|_{p,V_p} = 1$ and by weak lower semicontinuity $\|\nabla_{\mathbb{H}} u\|_2 \le c_r(N, p)$ and thus, necessarily, $\|\nabla_{\mathbb{H}} u\|_2 = c_r(N, p)$. Consequently, u is a minimizer. Furthermore, by weak convergence and convergence of the norm, we have $u_n \to u$ in $H_r^1(\mathbb{H}^N)$.

3. Finally, let $p = \infty$. Let $(u_n)_{n\in\mathbb{N}}$ be a minimizing sequence for (38), namely such that $\|\nabla_{\mathbb{H}} u_n\|_2^2 \to c_r(N, \infty)$, $|[g_t u_n](r_1)| \le 1$ for all $t > 0$ and there is a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that $[g_{t_n} u_n](r_1) \to 1$. Let $w_n = g_{t_n} u_n / [g_{t_n} u_n](r_1)$. Then we have $\|\nabla_{\mathbb{H}} w_n\|_2^2 \to c_r(N, \infty)$, $|[g_t w_n](r_1)| \le 1 + o(1)$ for all $t > 0$ and $w_n(1) = 1$. Then there is a $w \in H_r^1(\mathbb{H}^N)$ such that, on a renamed weakly convergent subsequence of (w_n), $w_n \rightharpoonup w$. By the compactness of the local Morrey embedding, $1 = w_n(r_1) \to w(r_1)$, while $|[g_t w](r_1)| \le 1$ for all $t > 0$. By weak semicontinuity of the norm, $\|\nabla_{\mathbb{H}} w\|_2^2 \le \liminf \|\nabla_{\mathbb{H}} w_n\|_2^2 = c_r(N, \infty)$. Thus w is a minimizer for (38), $\|\nabla_{\mathbb{H}} w_n\|_2 \to \|\nabla_{\mathbb{H}} w\|_2$, and, consequently, $w_n \to w$ and $g_{t_n} u_n \to w$ in the norm of $H_r^1(\mathbb{H}^N)$. Furthermore, we have

$$\begin{aligned}
\inf_{u(r_1)=1} \|\nabla_{\mathbb{H}} u\|_2^2 &\le \inf_{u(r_1)=1,\ \sup_{t>0}|[g_t u](r_1)|=1} \|\nabla_{\mathbb{H}} u\|_2^2\\
&= \inf_{\sup_{t>0}|[g_t u](r_1)|=1} \|\nabla u\|_2^2 = c_r(N, \infty).
\end{aligned} \tag{46}$$

Indeed, the inequality in the relation above is trivial, and the equality to the right of it follows from the trivial inequality

$$\inf_{u(r_1)=1,\ \sup_{t>0}|[g_t u](r_1)|=1} \|\nabla_{\mathbb{H}} u\|_2^2 \ge \inf_{\sup_{t>0}|[g_t u](r_1)|=1} \|\nabla_{\mathbb{H}} u\|_2^2 = c_r(N, \infty)$$

and the fact that our function w satisfies the additional constraint in the left-hand side.

Consider now the infimum in the left-hand side of (46). It is necessarily attained on a function which is harmonic on $(0, r_1)$ and on (r_1, ∞), which, by the requirement that its squared gradient is integrable, equals necessarily to the function (33).

We conclude that any minimizing sequence for (38) admits a renamed subsequence and a sequence of positive numbers (t_n) such that $g_{t_n} u_n \to w$ in the norm of $H_r^1(\mathbb{H}^N)$.

Furthermore, if $\tilde{w}$ is any minimizer for (38), the constant minimizing sequence $(\tilde{w})_{n\in\mathbb{N}}$ admits a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that $g_{t_n} \tilde{w} \to w$ in the $H_r^1(\mathbb{H}^N)$-norm. Then necessarily $t_n \to t$ with some $t > 0$ and $\tilde{w} = g_{\frac{1}{t}} w$. □

As a consequence of the profile decomposition we have the following compactness result. The intersection space below is assumed to have the scalar product that is a sum of scalar products of constituent spaces.

Theorem 5.4 *Let $N \geq 3$, $p \in (2, 2^*)$, and let*

$$W_p = V_{2^*}^{1-\frac{N}{2}(1-p/2^*))}. \tag{47}$$

Then the embedding $H_r^1(\mathbb{H}^N) \cap L^2(\mathbb{H}^N, \mathrm{d}V_{\mathbb{H}}) \hookrightarrow L^p(\mathbb{H}^N, W_p \mathrm{d}V_{\mathbb{H}})$ (and, therefore, the embedding $H_r^1(\mathbb{H}^N) \cap L^2(\mathbb{H}^N, \mathrm{d}V_{\mathbb{H}}) \hookrightarrow L^p(\mathbb{H}^N, \mathrm{d}V_{\mathbb{H}})$) is compact.

Note that the exponent in the right-hand side of (47) is positive, so the weight W_p is bounded away from zero and goes to infinity at $r = 1$.

Proof Assume without loss of generality that $u_n \rightharpoonup 0$ in $H_r^1(\mathbb{H}^N) \cap L^2(\mathbb{H}^N, \mathrm{d}V_{\mathbb{H}})$ and consider the subsequence of (u_n) given by Theorem 5.3. Note that for each n such that $(j_k^{(n)})_k$ has a subsequence convergent to zero, the corresponding profile $w^{(n)}$ will be zero. Indeed, taking into account that $r^2 V_2(r)$ is bounded away from zero (since it follows that $\frac{f(r)(1-r^2)}{G(r)} \geq \frac{C}{r}$ from (16) and (17) for $N \geq 3$, and from (14) for $N = 2$), we have

$$\int_{\mathbb{H}^N} |w^{(n)}|^2 \mathrm{d}V_{\mathbb{H}} \leq C \int_{\mathbb{H}^N} r^2 |w^{(n)}|^2 V_2 \mathrm{d}V_{\mathbb{H}} \tag{48}$$

$$= \lim_{k\to\infty} \int_{\mathbb{H}^N} (G^{-1}(1/j_k^{(n)} G(r))^2 u_k|^2 V_2 \mathrm{d}V_{\mathbb{H}} \to 0. \tag{49}$$

Therefore, all nonzero terms $g_{j^{(n)}} w^{(n)}$ in the profile decomposition of u_k will correspond to sequences $(j_k^{(n)})_k \to \infty$, which implies, by a calculation similar to the one above, that $g_{j^{(n)}} w^{(n)} \to 0$ in $L^2(\mathbb{H}^N)$. Thus the remainder in (45) is bounded in $L^2(\mathbb{H}^N)$. Consequently, u_k vanishes in any norm defined by the interpolating between $L^2(\mathbb{H}^N)$ and $L^{2^*}(\mathbb{H}^N, V_{2^*} \mathrm{d}V_{\mathbb{H}})$ by means the Hölder inequality, that is, in the $L^p(\mathbb{H}^N, W_p \mathrm{d}V_{\mathbb{H}})$-norm. □

We add now a more elementary compactness result.

Theorem 5.5 *Let $p \in [1, \infty]$ and let $V(r)$ be a measurable function such that, for some $C > 0$ and all $r > 0$, $|V(r)| \leq C V_p(r)$. If*

$$\lim_{r\to 0} V(r)/V_p(r) = \lim_{r\to 1} V(r)/V_p(r) = 0, \tag{50}$$

then the embedding $H_r^1(\mathbb{H}^N) \hookrightarrow L^p(\mathbb{H}^N, V \mathrm{d}V_{\mathbb{H}})$ is compact.

The proof of this statement is entirely similar to that for Theorem 1.1 in [12], is left to the reader. Note that this result is sharp in the sense that if $V = V_p$, then the embedding is not compact, due to invariance of both norms with respect to the non-compact group D.

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Appendix: case $N = 2$

The case $N = 2$ of the present paper has been already handled by [3], which did not mention hyperbolic geometry, but studied, inter alia, the radial Sobolev space $H_{0,r}^1(B)$ of the unit disk.

However, since the latter space is isomorphic to $H^1_r(\mathbb{H}^2)$, all the results of [3] for $N = 2$ allow an immediate interpretation in terms of hyperbolic geometry. We cross-reference here the results in [3] for $N = 2$ with the results in this paper.

- The scaling (1.6) in [3] is the same as (10) with $G(r) = \log \frac{1}{r}$.
- The scale-invariant weights V_p in [3, formula (1.5)], coincide with the definition (12).
- The radial embedding of Theorem 3.6 follows by interpolation between Leray inequality and the pointwise estimate of [3], Corollary 2.2.
- Cocompactness of the embedding $H^1_r(\mathbb{H}^2) \hookrightarrow L^p(\mathbb{H}^2, V_p \mathrm{d}V_{\mathbb{H}})$ for $p \in (2, \infty]$ (Theorem 5.1 for $N = 2$) was proved in [3], Lemmas 3.3 and 3.4.
- Theorem 4.2 in [3] is based on a false statement that the weight V_p is decreasing (the authors have asked the journal to publish an erratum). Its argument is still valid, however, in restriction to radial functions, and gives the case $N = 2$ of Theorem 3.7 in the present paper.
- The profile decomposition of Theorem 5.3 for the case $N = 2$ is [3, Theorem 5.1].

Remark 5.6 Moser–Trudinger inequality, or, more precisely, its refinement for hyperbolic spaces proved in [4,20], can be also identified as a two-dimensional case of a general inequality for $N \geq 2$,

$$\sup_{u \in H^1(\mathbb{H}^N), \|\nabla_{\mathbb{H}} u\|_2 \leq 1} \int_{\mathbb{H}^N} ([G^{-1}(\omega_{N-1} u^2(x))]^{-2} - 1)^{N/2} \mathrm{d}V_{\mathbb{H}} < \infty. \tag{51}$$

For $N \geq 3$, however, the asymptotics of the integrand, as a function of u, are, by (16),(17), $\sim u^2$ at zero and $\sim |u|^{2^*}$ at infinity. Therefore, (51) is a consequence of the embedding of $H^1(\mathbb{H}^N)$ into $L^2(\mathbb{H}^N) \cap L^{2^*}(\mathbb{H}^N)$ from [19], which in turn is a consequence of (27).

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